CONNECTIONS BETWEEN THEORY AND RESEARCH FINDINGS: THE CASE OF INEQUALITIES

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Existing literature has shown a growing interest in examining students' conceptions of mathematical notions, their difficulties when dealing with mathematical tasks and possible reasons of these difficulties. In this study, we focus on students' ways of thinking when solving algebraic inequalities. The present study was designed to extend the existing body of knowledge regarding students' understanding and their difficulties when solving algebraic inequalities. More specifically, we focus on students' solutions to algebraic inequalities including a parameter.

In this paper we adopt two different theoretical models suitable for understanding students' difficulties from different perspectives. Consequently, we suggest possible implications for teaching. More precisely we use Fischbein's theory dealing with the notions of intuitive, formal and algorithmic knowledge in one's mathematical performance (e.g., Fischbein, 1993) and a theoretical model referring to the distinction between sense and denotation of algebraic expressions (Arzarello, Bazzini and Chiappini, 1993, 2000: in short ABC from here on).

While Fischbein's theory is mainly centered on the student's performance, the ABC model is centered on the very meaning of algebraic expressions; although their starting point is different, the two theories share as common objective the analysis of students' difficulties, their roots and possible overcoming.

According to Fischbein, formal knowledge is based on propositional thinking. It relates to rigor and consistency in deductive construction, being free of the constraints imposed by concrete or practical characteristics. Intuitive knowledge is a kind of cognition, which is accepted directly and confidently as being obvious, imparting the feeling that no justification is required. Algorithmic knowledge is the ability to use theoretically justified procedures. Each of the three components plays a vital part in students' mathematical performance, but since they are usually inseparable, the relations between them are not less significant. Fischbein explained that "sometimes, the intuitive background manipulates and hinders the formal interpretation or the use of algorithmic procedures" (ibid. p. 14). Consequently, he identified and investigated with his colleagues a number of algorithmic models related to various mathematical operations, such as subtraction of natural numbers, and methods of reduction in processes of simplifying algebraic expressions (e.g., Fischbein & Barash, 1993).

As far as the ABC model is concerned, the authors distinguish between sense and denotation of an algebraic expression. The denotation is the object to which the expression refers, while the sense is the way in which the object is represented. There are expressions having the same denotation but different senses: for example, the expressions $4x + 2$ and $2(2x+1)$ express different rules (senses) but denote the same function. In the case of equations and inequalities, denotation is the set of values which make the equation (inequality) true, i.e. the truth set. So, the same truth set can be represented in different ways (i.e. senses), as being attached to different equations (inequalities). Likewise, the same expression can have different senses, for example in relation with the domain of definition. Parametric expressions need special attention because they usually represent a set of numerical expressions and, as so, denotation is concerned with a range of values.
We analyze Italian and Israel secondary-school students' performance when solving algebraic inequalities including a parameter.

In particular, we focus on the following questions:

1. What intuitive ideas and what algorithmic models can be identified in Israeli and Italian secondary school students' solutions to algebraic inequalities?
2. Are these ideas and models connected to the problem of identifying denotation in the given expressions?

192 Italian and 210 Israeli high school students participated in this study. All participants were 16-17 year old who planned to take final mathematics examinations in high school. Italian and Hebrew versions of a 15-task questionnaire were administered to the students. Here we focus on three tasks dealing with "dividing an inequality by a not-necessarily-positive factor". Two tasks ask to judge statements regarding parametric inequalities, and the third task, requires to "solve" a parametric inequality.

Task I: Examine the following claim: for any a in R, $a\cdot x < 5 \Longrightarrow x < 5/a$
Task II: Examine the following statement: for any $a\neq 0$ in R, $a\cdot x < 5 \Longrightarrow x < 5/a$
Task III: Solve the inequality: $(a-5)\cdot x>2a-1$ x being the variable and 'a' a parameter.

The three tasks require, more or less implicitly, the recognition of the denotation of the given expressions. More precisely, such recognition is needed to judge the sense of Task I and Task II, while in Task III one has to recognize that denotation depends on the parameter a.
An initial examination of the Italian and Israeli participants' written solutions to Task I revealed almost no intuitive, erroneous ideas. Most of the students correctly rejected the statement: a R $a\cdot x<5 \Longrightarrow x<5/a$, and based this rejection either on a comprehensive analysis of the sign of a, or on a single counterexample. In the latter case, many students used zero as their counterexample, and most of them, correctly stressed that one counterexample is sufficient for the refutation of a proposition.

Had we stopped here, we might have assumed that most students have a good formal understanding of such parametric inequalities. However, an examination of their responses to tasks II and III revealed that this was not the case. The students' solutions to the latter two tasks, and their explanations in the interviews, clearly pointed to their intuitive grasp of inequalities as being "similar to", "the same as", "a certain type of" or analogous to equations. That is to say, equations were found to serve as a prototype in the algorithmic models of solving inequalities. This algorithmic model had two appearances: (1) "do the same operation with the same numbers on both sides"; or (2) "exclude the possibility of zero in the denominator, and then, do the same operation with the same numbers on both sides".

Such confusion between the notion of equation and that of inequality clearly reveals difficulties in identifying denotation. For those students, the given algebraic expression activates only a procedural sense, disconnected from the denotation which stands behind (see also, Bazzini & Tsamir, 2001; Tsamir, Almog, & Tirosh, 1998; Tsamir & Bazzini 2001; 2002).

We would like to suggest such tasks to be presented and discussed in class, in order to promote students' awareness of their intuitive ideas and the resulting algorithmic model they intuitively use, together with a reflection on what solving an inequality really means. The relevant role of denotation should emerge consequently.

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