LEARNING DIFFICULTIES BEHIND THE NOTION OF ABSOLUTE VALUE

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Abstract. *Teaching experiences and research on learning have convinced us that the notion of absolute value, while does not present relevant difficulties when used on numbers, originates errors and misconceptions when used on letters.*

In order to carry out a longitudinal study on these errors we administered a questionnaire focused on the absolute value to students of three different school levels (ages: 14, 17, 19 years). The analysis of the results of the questionnaire shows that this notion hides different types of obstacles which seem to be in relation with many of those studied by different researchers in the field of algebra learning. In particular, the possibility of verifying the persistence of certain errors at different ages allows to point out which "wrong beliefs" rooted in student's mind are difficult to eradicate.

Introduction

Usually absolute value is not object of particular attention in school practice: it is considered not a real concept, but just a stenography for mathematical situations centred upon positive and negative numbers. On the other hand the experience of many teachers of secondary school suggests, as Duroux (1983) does, that absolute value is truly the "bête noire" in the manipulation of symbols and computations. For example one of the difficulties we observe in the students' learning of the concept of limit is the use, in the formal definition given in Weierstrass style, of the absolute value.

The students' mistakes in manipulating absolute value confirm the opinion of Hart (1981) that "the teaching algebra may be perceived by students as an initiation into rules and procedures which, though very powerful (and therefore attractive to teachers), are often seen by students as meaningless" and the opinion of Booker (1987) that "the use of symbols has little or no meaning for the students who have to manipulate them".

In order to study the difficulties hidden in this notion and to put them in the global frame of algebra learning difficulties we carried out the research described in this report.

Some features of the didactic practice as for absolute value

In school practice absolute value may be seen at different levels of conceptualization which concern the *arithmetic* or the *algebraic domain.* In the following we will fix our attention on this last domain, since we have formed the opinion that in the arithmetic field the absolute value does not present significant problems.

As a matter of fact the first introduction of absolute value happens in the *arithmetic domain*: numerical examples of the type |+7|=+7, |-7|=+7 are given. More often the numerical examples are presented in the following way: |7|=7, |-7|=7, according to the convention that numbers without sign are positive. At this level we observe that the notion of absolute value is perceived by students as a kind of device that eliminates the sign before the given number. When this notion is enriched with the geometrical meaning (that to say, given the real line the absolute value of a number is the distance of the point with this number as abscissa from the origin) the fact that any number becomes positive through absolute value results more emphasized and is more explicitly perceived by students.

We have administered preliminary tests (which are not object of the present report) concerning this arithmetic domain to students and, in the same time, we have interviewed teachers on their school experiences concerning the subject. Our conclusion is that at this arithmetic stage, for the fact that it concerns the manipulation of absolute value applyed only on numbers, there are no particular difficulties. This confirms the common opinion of the researchers I quote in this report that the use of numbers helps the understanding of symbols and operations.

The *difficulties arise in passing from the arithmetic to the algebraic domain*, when student has to face the difficulty of formalizing arithmetical patterns.

Bell (1988), Booker (1988) and Kieran (1988) have analysed the difficulties linked with the generalization in passing from arithmetic to algebra. These difficulties mainly concern the meaning of the operations and of the conventions in procedures different from those of arithmetic: errors in manipulations are the more immediate consequence. Moreover, all these authors, Olivier (1988) and, in particular, Rosnick (1981), Wagner (1983) recognize as a common cause of difficulty in working in algebra the use *of letters instead of numbers,* that is to say the intervention in algebraic problems of *variables* with the related problems of interpreting the meaning of letters.

As for absolute value, in the algebraic domain the definition is usually given in the standard way, that to say $|x|=x$ if $x\geq 0$, $|x|=-x$ if $x<0$, without particular comments (apart the illustration of the geometrical meaning). The only formal change in respect to the definition given in arithmetic is to deal with letters instead of numbers. The notion is perceived by students according to two orientations, which recall the distinction illustrated by Sfard (1987) between the two conceptions (operational and structural) of mathematical notions: in certain contexts absolute value is considered *as an operator from R to* R^{+} *{0}*, in other situations *as a restriction of the field R to* R^{+} *{0}.*

This perception is unconscious and even if usually is made explicit neither by teachers nor by students affects the learning of the subject and the student' performances. Taking into account these two conceptions and the consequent difficulties allows to enlighten the hidden educational and epistemological obstacles (Duroux).

For example, it is clear that the operational conception leads to the errors observed in secondary level algebra by Becker (1988) concerning domain and image of functions and individuation of variables and parameters. The structural conception leads to the errors linked with the recognition and the use of structures considered by Kieran (1988).

The considerations about the lack in understanding logical operators (and, or, implies,...), quantification, distinction between proof and counterexample pointed out in the work of Dubinsky E., F. Elterman and C.Gong (1988) fit to the subject we are studying as well.

A questionnaire of investigation

In order to render explicit and discuss the intervention of all the above factors in the students' manipulation of absolute value we have prepared a questionnaire with the aim of testing the students' performances in relation to the difficulties pointed out in the previous considerations. The questions are divided in groups mainly focused on : - *functions* (1, 2) - *geometrical representation* (3, 10) -

letters and variables (5, 8, 11) - *literal calculation* (4, 6, 7) - *the usefulness of absolute value in solving problems* (9).

Of course, as we will see in the following discussion, it is not possible to make a rigid separation of subjects and difficulties: elements characterizing a group may intervene as secondary elements in other groups. Certain other elements, while not explicitly mentioned intervene in certain group of questions: for example, the concepts related to identities and equations analysed in the work of Herscovics N. and C. Kieran (1980) intervene in questions 4, 5, 6 and 7.

The questionnaire was administered to 273 students of the following three categories:

(a) students of the age of 14 (first year of upper secondary school)

(b) students of the age of 17 (fourth year of upper secondary school)

(c) students of the age of 19 (first university year: course of mathematics and of earth science).

In theory all the students target of the experiment were able to answer the questions; in practice the youngest ones had some handicaps. More in details, we consider that students of the category (a) know a definition of absolute value usually centred upon numerical examples and sometimes its geometrical meaning of distance from origin as well. They have little experience in solving problems in which the result need of discussion involving absolute value, since until the school level in question usually the data of problems are mainly numerical and the domain of problem is mainly arithmetic. Even if they deal with manipulations of literal expressions and know the basic notions concerning the Cartesian plan, their command in manipulating the elements in question is not completely achieved.

Students of the category (b) are supposed to have a good command in numerical and literal manipulations, including solution of equations, and to have been introduced to the concept of function. At the school level in question students begin to apply formally the hypothetical-deductive reasoning and prove some theorems.

Students of the category (c) have accomplished their pre-university mathematical curriculum and have chosen a career scientifically oriented. Moreover they know the basic elements of calculus, in particular the rigorous definition of limit and continuity. Both these last two categories have the occasion of working with absolute value in solving theoretical and practical problems.

The allowed time for answering the 11 questions was reduced of one third for the university students, taking into account that they are selected and in theory have a better command in doing mathematics.

The check at different school levels allows to compare the student' performances when : (a) they are passing from the arithmetic to the algebraic domain, (b) they are hardly working on algebraic manipulations, on solving equations and on elementary functions, (c) the notions and concepts should be in theory well settled in student' mind. The aim of the longitudinal experiment is to check how much working in mathematics affects the *real* understanding of mathematics. In particular, we wish to verify the "degree of certitude" of certain schemas that students construct in their minds and that, sometimes, contain what Furinghetti and Paola (1988) call "wrong beliefs".

Results of the questionnaire and comments

QUESTIONS INVOLVING THE CONCEPT OF FUNCTION

Question 1: For which values of x   R is |x|<0 ?

Question 2: Fixed x   R, how many numbers are represented by |x -1|?

Percentages of correct answers:

Question 1:	14 years: 53%	17 years: 70%	19 years: 91%
Question 2:	14 years: 32%	17 years: 6%	19 years: 49%

The most common wrong answer to question 1 is "$|x|<0$ if $x<0$".

If one takes into account that virtually all the students in question know a definition of absolute value and, moreover, the standard definition of absolute value ($|x|=x$ if $x\geq 0$, $|x|=-x$ if $x<0$) was also written on the top of the sheet containing the exercises, this kind of results may be considered curious, but it is not unexpected for the researchers in mathematics education.

The most immediate and spontaneous interpretation of this fact is to ascribe the cause of the error to the student's difficulty in interpreting algebraic symbols. That is the opinion expressed by Matz (as quoted in Wagner (1983)) who asserts that "the sign perversity of literal symbols makes one of the standard definitions for absolute value ($|x|= x$ if $x\geq 0$, $|x|=-x$ if $x<0$) virtually incomprehensible to many students". Analogously, the assertion in Burton (1988) that "students are often unable to read an algebraic sentence in the sense of extracting meaning from it.[...] even in the sense of *seeing* all of it." may be a further explanation of the error.

In our opinion a satisfying explanation is the following: in the standard definition absolute value appears as an operator applied on x and so the standard wrong answer $|x|<0$ if $x<0$ to question 1 is originated by the misconception in individuating domain and image of a function. This opinion is supported by the common output we have from university students, when they have to draw, for example, the graph of $f(x)=|\sin x|$ or similar graphs.The usual student's answer in this kind of exercises, is to draw the correct curve, but only on the positive side of axis *x*.

Other difficulties related to absolute value perceived as an operator come from the fact that it is a kind of piecewise function. On this subject the works of Marcovits Z., B. Eilon and Bruckheimer M. (1986) and of Tall and Vinner (1981) point out that this kind of functions present difficulties. We think the difficulty hidden in the standard definition of absolute value (and in the piecewise functions) is the presence of the logical operations "or" ($|x|= x$ or $-x$) and "implies" (if $x\geq 0$, if $x<0$): very often student tends to interpret "or" as "and" and does not care about the "if"..."then". So they consider in the same domain different images.

Without any doubt if these logical operations are rendered more explicit, student's understanding of the notion is improved. This is the case of the definition of absolute value presented as an introductory exercise to programming: the effort of writing in a precise language the conditions ($|x|= x$ ...) enunciated in the definition leads to a clarification of the mathematical terms intervening.

The previous considerations concern the results as for question 2 as well.The most common wrong answer is "$|x -1|$ represents two numbers" and confirms the misunderstanding between domain and image, variable dependent and independent. Moreover this wrong answer is linked to the difference between working with numbers or with letters. We have observed at the beginning of the present report the students in question are able to answer correctly to "|7|=...?", now we observe that the same question proposed in terms of letters presents relevant difficulties.

Another category of error concern the word "fixed", since many wrong answers are of the type "|*x* -1| represents infinite numbers, depending on the value of *x* ". In our opinion this error is related to the ambiguity, present especially in pre-university courses, in the distinction between variables and parameters, which has provoked the bad scores of students aged 17. We suppose also that in the same time the inability stressed by Burton (1988) to "read an algebraic sentence" and to "see all of it" plays a relevant role in neglecting the word "fixed".

QUESTIONS INVOLVING GEOMETRICAL REPRESENTATION

Question 3. Mark on the real line two points A of abscissa a, B of abscissa b, a<b, such that distance OA >distance OB.

Question 10. Given a system of Cartesian axes, draw the graph of $|x|$*, x R. (Hint: consider for x integer values between -5 and 5).*

Percentages of correct answers:

Question 3:	14 years: 60%	17 years: 93%	19 years: 83%
Question 10:	14 years: 51%	17 years: 92%	19 years: 71%

The good results as for these items prove the greater appeal of the geometrical language in respect to the algebraic one. In particular we observe that the question 3 is the geometrical translation of the definition which created difficulties to students in question 1.

QUESTIONS INVOLVING LETTERS AND VARIABLES

Question 5. How many solutions has the equation $|x|=-a$*, for a R*?

Question 8. Is it always true that the fraction $|a|/-a = -1$, *for a R,* $a\neq 0$?...*Why?*

Question 11. *Do exist x such that the equality* $\sqrt[4]{x^2}=\sqrt{-x}$*, x R is true?*

Percentages of correct answers:

Question 5:	14 years: 2%	17 years: 17%	19 years: 40%
Question 8:	14 years: 56%	17 years: 69%	19 years: 60%
Question 11:	14 years: 39%	17 years: 57%	19 years: 39%

The three items are in relation to the common difficulty of dealing with letters and variables, above hinted in discussing question 2.

The concept image of the letter without sign considered always as positive number is so strong that the majority of students rejects the idea that *-a* or *-x* may be negative. Even those students who have rightly answered the question concerning the meaning of absolute value, show this inconsistency in the interpretation of letters, confirmed by the fact that the great majority of the correct answers are justified by numerical checking and not by the general statement that in the real field *-a* (or *-x*) represent s positive numbers if *a* (or x) are negative.

The results concerning question 5 are the worst of all the questionnaire: as a matter of fact in this item the difficulty of letters and the concept of equation, its solution, difference between parameters and variables as well intervene. In particular the general bad results for all the three categories of students in this question is caused by the obstacle peculiar of absolute value: student has the impression of dealing with an algebraic equation of *first degree* ($|x|=-a$, for *a* R in which *x* has degree *1*) which has *two* solutions and this fact contrasts with the rule connecting the order (here supposed) of an algebraic equation and the number of the solutions. So the symbol of absolute value reveals a further ambiguity which , perhaps, could be removed by opportune geometrical representations.

It may interesting to observe the idea that $\sqrt{-x}$ is a writing always without meaning is so firmly rooted in student's mind that the wrong answers of university students are enriched with the explanation: "the equality $\sqrt[4]{x^2}=\sqrt{-x}$, is never true in the real field, but it may be true in the complex field" (according to student's feeling, the complex field is the domain in which anything about roots can happen). Between other things, this explanation proves that the use of the symbol $\sqrt[n]{x}$ made in many textbooks is misleading, both in real and in complex field.

QUESTIONS CONCERNING LITERAL CALCULATION

Question 4: Draw on the real line the interval of x such that $|x|<3$

Question 6: Is it true that $|x-1|=x-1$, *for* x R *and* $x>0$?

Question 7: Is it true that $|1-x|=1+x$, *for* x R *and* $x>1$?

Percentages of correct answers:

Question 4:	14 years: 39%	17 years: 80%	19 years: 75%
Question 6:	14 years: 25%	17 years: 57%	19 years: 69%
Question 7:	14 years: 54%	17 years: 72%	19 years: 69%

These three items are aimed to check the command in literal calculations: no surprise that students accustomed to perform a great deal of calculations and to apply them in exercises of calculus (limit, continuity and so on) reach better results. The youngest students fail the item 4, since they neglect to consider the solution -3 of the equation $|x|=3$. Nevertheless, since this item is more depending on routine exercises and refers to the geometrical representation, all the three categories of students have a better result in comparison with the analogous item 5, which is more in connection with the understanding of letters and variables (*-a appears instead of 3).*

As for the items 6 and 7 the majority of correct answers is obtained by checking with suitable numbers and, in the case, giving a counterexample. But, even if, at a first look, the two items seem to present difficulties of the same level, the performances of students are different, because the strategy of solution by checking numbers works in the item 6 only if the number are chosen in the interval [0,1). If the student mistakes this choice, he has to apply the definition of absolute value and to do the relative calculations. In this case a common mistake is the wrong application of the "implies" as emerging from this common student's explanation: "the equality $|x-1|=x-1$ is true for $x\geq1$ and so for $x>0$ as well".

Moreover other students' explanations shows once more that misunderstandings in considering the domain of an operator persist: it is obvious for students to discuss $x>0$, while it has to be considered where $x-1>0$.

The relatively good scores in this group of questions are a confirmation that in case of algebraic manipulations students, even in absence of a real and critical understanding, are able to perform certain exercises (by rote).

QUESTION CONCERNING THE USEFULNESS OF ABSOLUTE VALUE IN SOLVING PROBLEMS

Question 9: In a right-angle triangle a side is $2\sqrt{a}$ *cms and the hypotenuse is* $a+1$*cms,* $a>0$.

Find the length of the other side.

Percentages of correct answers:

Question 9: 14 years: 30% 17 years: 30% 19 years: 17%

All the students in question except three applyed correctly the Pythagoras' theorem, the wrong answer consist in having neglected the condition that the solution is the length of a segment and has to be positive, so they write a-1 instead of the correct $|a\text{-}1|$.

The bad results in this last item, in particular as for university students, suggest some considerations concerning the teaching of this neglected topic.

Absolute value is introduced without valid motivations and students accept it, as it happens for other algebraic manipulations (see once more Hart (1981)), passively and without grasping neither the effective meaning nor the usefulness. They learn by rote algebraic manipulations, but remove as soon as possible the underlying concepts. Probably a definition introduced with adequate motivations (in approximation problems, in solving problems of physics and so on) would help student in constructing good schemas and avoiding the "wrong beliefs" that are cause of errors.

Concluding remarks

This research may have further developments in the direction of analysing more in details the intervention of hidden obstacles in constructing the concept of absolute value and of studying remedies in teaching. Cognitive science may be a good help, as hinted by Wenger for other algebraic subjects (1988). We would like to focus two points emerging from the percentages we have shown in this report.

Firstly we observe that only in four questions (1, 2, 5, 6) on the totality of 11 the university students reach the best results, in two (9, 11) their scores are at the same level or worse of those of the youngest students. This fact hints that: knowledge constructed on precarious grounds is forgotten after a while and in the same time the wrong beliefs rooted in student's mind (see, for example, the sign not incorporated in the letter) show a persistence which appears not to be dependent on the quantity of learned mathematics.

Since we ascertained that conflicts are present in student's mind as for absolute value (see questions 1 and 5), teachers may consider these conflicts (as Olivier (1988) suggests) a good starting point for their work of restyling the notion in question .

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