WHEN TABLES BECOME FUNCTION TABLES[1]

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This study explores third-grade students' strategies for dealing with function tables and linear functions as they participate in activities aimed at bringing out the algebraic character of arithmetic. We found that the students typically did not focus upon the invariant relationship across columns when completing tables. We introduced several changes in the table structure to encourage them to focus on the functional relationship implicit in the tables. With a guess-my-rule game and function-mapping notation we brought functions explicitly into discussion. Under such conditions nine-year-old students meaningfully used algebraic notation to describe functions.

Most mathematics educators, ourselves included, tend to view data tables as function tables. But what about the students? Are they learning about functions when they fill out tables? What does it take for third grade students to treat multiplication tables, for example, as function tables? Can they use and understand algebraic notation for representing linear functions? What sorts of activities involving tables might encourage young students to focus on functional relations?

Students begin to deal with (linear) functions and (constant) rates long before they make any sense of an expression like $y = \mathrm{m}x + \mathrm{b}$. Certain curriculum materials embody these relations without making them explicit in algebraic notation. A multiplication table, for instance, might be thought of as an embodiment of the expression y = mx, where x and y are integers along the margins and m corresponds to the number in the expression "times <m> table". The question we raise here is whether children as young as nine years of age can understand functions and algebraic notation for functions.

The studies by Davydov and colleagues (1991/1969) showed that young students were able to use and understand algebraic notation such as y = 5x+12. However, in their studies x and y stand for unknowns. We know of no evidence from their work suggesting that students thought of the notation as expressing a multitude of ordered pairs and hence functions; and this is unlikely since problems were invariably constrained in such a way as to require that x and y take on single values. To make sure that students are contemplating *multiple* input and output values, so to speak, it is useful to consider situations where the same function is applied repeatedly. Let us look at a sales context first; then we will move to the issue of how it is embodied in a function table. These are precisely the conditions underlying our work with third graders as we explore how, in keeping with the U.S. National Council of Teachers of Mathematics (2000) Standards, algebraic reasoning and notation can become part of the elementary school curriculum.

In our approach, we treat algebra as a generalized arithmetic of numbers and quantities. Accordingly, we view the transition from arithmetic to algebra as a move from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures, from computing numerical answers to describing relations among variables. This requires providing a series of problems to students, so that they can begin to

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note and articulate the general patterns they see among variables. Tables play a crucial role in this process as they allow one to systematically register diverse outcomes (one per row) and look for patterns in the results.

Our initial steps were inspired by the findings of everyday mathematics studies. When computing the price of a certain amount of items, street sellers usually start from the price of one item, performing successive additions of that price, as many times as the number of items to be sold (Nunes, Schliemann, & Carraher, 1993; Schliemann et al., 1998; Schliemann & Nunes, 1990). If we try to understand their procedure in terms of displacements in a function table, they work down the number column and the price column, performing operations on measures of like nature, summing money with money, items with items. Vergnaud (1983) describes this strategy as a "scalar approach". In contrast, a functional approach presumably relies upon relationships between variables, often variables of different natures. The latter focuses on how one variable changes as a function of the other variable.

But when we look closely at street sellers' strategies we realize that they establish a correspondence of values across measure spaces before proceeding to the next case. The flow of thought proceeds from one measure space to the other, row by row. This is illustrated by the following solution by a coconut seller to determine the price of 10 coconuts at 35 *cruzeiros* each:

> "Three will be one hundred and five; with three more, that will be two hundred and ten. [Pause]. I need four more. That is… [Pause] three hundred and fifteen… I think it is three hundred and fifty." (Nunes, Schliemann, & Carraher, 1993, p. 19).

The street sellers' approach indeed involves a linking of a unique y-value to each value of x. It therefore captures the essential idea of a function and can constitute a meaningful and efficient strategy to solve missing value proportionality problems. In school young children also seem to prefer using scalar solutions (Kaput & West, 1994; Ricco, 1982). Scalar solutions can be a good start for understanding functions. But they are limited in scope and typically do not allow for broader exploration of the relationships between the two variables (Schliemann & Carraher, 1992, Carraher & Schliemann, 2001).

In the classroom study here reported we will look at some specific examples of how third-grade students' emerging understanding of functional relations draws upon and at the same time departs from their previous strategies for dealing with quantities and number relations.

The data come from a broader study aimed at understanding and documenting issues of learning and teaching in an "algebrafied" (Kaput, 1995) arithmetical setting (see Carraher, Brizuela, & Schliemann, 2000; Carraher, Schliemann, & Brizuela, 2000, 2001). Our goal was to help children build an understanding of multiplication from an algebraic point of view and as a functional relationship. To reach this goal, we designed activities that aimed at shifting the focus from scalar relations to functional relations and to general algebraic-type notational representation. Through a discussion of children's difficulties and successes, as they participate in these activities, we will explore some of the issues they face in trying to move from their intuitive approaches to a functional approach and from computations to generalizations.

The Study

We worked with a classroom of 18 third-grade students at a public elementary school in the Boston area, serving a diverse multiethnic and racial community. During the school year, we met with them once a week for a period of ninety minutes. The first six meetings were dedicated to additive relations (see Carraher, Brizuela, & Schliemann, 2000). In the seventh week, as the children were working on learning the multiplication tables, we started working on multiplicative relations. Our challenge at this point was to design situations that would allow children to understand multiplication as a functional relationship between two quantities or numbers.

We used what we knew about street sellers and young children's strategies to solve price problems as a point of departure. From our perspective, the organization of data for two related quantities in a table would provide the opportunity for children to use their own scalar strategies but would also allow us to explore with them the implicit functional relationships between two variables. The sequence of tasks we designed was presented and discussed over two weekly meetings (classes 7 and 8). The first two tasks were part of class 7 and the other four were part of class 8.

Task 1: Filling out function tables

We began by asking children to fill out the table in Figure 1. Each child received a work sheet, but we suggested that they could work in pairs and discuss their solutions, helping each other.

Figure 1: The incomplete table

Mary had a table with the prices for boxes of Girl Scout cookies. But it rained and some numbers were wiped out. Let's help Mary fill out her table:

Boxes of cookies	Price
	$ 3.00
2	$ 6.00
3	
	$ 12.00
5	
6	
	$ 21.00
8	
9	
10	$ 30.00

Most of the students in the class first appeared to treat each column, items and price, as if they were separate problems. They would fill out column one by counting by 1's and column 2 by counting by 3's. Their approach leads to correct answers but does not involve them in thinking about the general relationships between price and items. A few children related the task to the multiplication tables they were memorizing and used the latter to fill out the second column in the table.

Task 2: Different ways to go from one number to another

The remainder of this class was dedicated to an activity where the children had to find different ways to operate on a number in order to get to another (e.g., "How do you get from 2 to 8?" and "How do you get from 3 to 15?"). This activity constituted an attempt to have children exploring the multiple relationships between two numbers in a pair. We hoped that this would later help them to focus on determining the relationship in a function table.

The first and most popular solutions were additive solutions such as: To get from 2 to 8 you "add 6 to 2" or "add 2, plus 2, plus 2." As discussions developed, children also used multiplication as alternative ways to get from one number to the other.

Task 3: Focusing on any number (N)

The following week we first presented children with a multiplication table similar to the one they had worked with, except for an added "Nth" row. Our goal here was to encourage children to think about the general relationships depicted in the table. They were asked to answer: What do you think the N means? What is the price if the number of boxes is N?

Again, children easily filled in the blanks by counting by ones in the first column and counting by threes in the second. David, the instructor, asked them to explain how they found the number that corresponded to 4 and one child responded that he added four threes. For the same question regarding the second row, one child explained that it was three times two and another that she had added 4 to 2. For the N^{th} row, one of the students, Sara, stated: "add 3 up; 11 times 3 equals 33; N probably stands for 11." Other children also considered that N was 11 and that the corresponding value in the second column was 33.

David explained that "N stands for anything." A child volunteered, "It could be any number." After discussion and examples, three children maintained 33 as a response in their worksheets, three left the cell blank, five adopted N+N+N or NNN as their response, and seven adopted the notation 3N or Nx3. One girl wrote on her work sheet the expression Nx3 followed by the equals sign: "Nx3=".

Task 4: Breaking the columns' pattern
After noting the predominance of column-by-column solutions, we decided to introduce breaks in the table sequence (Figure 2), thus hoping to draw children's attention to the functional relationship.

Figure 2: Filling out a table and generalizing to higher values and to N

2. Here is another table. Can you fill in the missing values?

X	Y
1	3
2	5
3	7
4	9
5	
7	
8	
9	
10	

20	

30	

100	

N	

This table was more demanding since it represented a function with an additive term ($x \rightarrow 2x + 1$). Children did not spontaneously focus upon the functional relationship and needed external help to complete the table. With help many were able to apply the rule and to complete the table.

Task 5: Developing a notation for the function
The next step was to focus on a general notation for the function. David wrote the rule nx2+1 on the board and worked with the whole class, assigning different values for n and computing the result. The same was done for 3n+2. He replaced n by different numbers, including zero and 1000, and children easily computed the output.

Task 6: Finding the rule from pairs of numbers

For the next activity, pairs of numbers were given and children were asked to find the rule that originated them. For the first trial of this new task, David wrote 3 and 6 as a first pair and 7 and 10 as a second pair. As the discussion below shows, the children found that they had to add 3 to the first number.

> David: Let's work backwards, we'll start from the numbers, and you tell me what the rule is. Can you do that?
> Student: Yes.
> David: All right. I'm going to start, I'm not going to tell you what the rule is.
> Student: You have to do it in pairs?
> David: Well, yes. Hold on. I'm going to start with three […] Then I'm going to go to there, to six [writes "3→6"]. OK, attention.
> Sara: Can I tell you the rule?
> David: OK, and now, if I start from seven, I'm going to go to 10 [writes "7→10"]..
> Sara: Can I tell you the rule?
> David: Does anybody have the rule figured out? If I start from five, I'm going to go to.
> Sara: Eight.
> David: Yes, I'm going to go to eight. I think somebody knows the rule! Jennifer! What were you thinking? What's the rule?
> Jennifer: Plus three?
> David: Yes! If I start from n, then I have to go to what?
> Students: (Inaudible).
> David: Three?
> Student: You have to add three.
> David: I have to add three to what?
> Student: To the n.
> David: Yes, to the n. So how am I going to write that down?
> Students: N plus three.
> David: Yeah! That's the rule!
> Students: Oh, we need something harder.

The children take the rule "N becomes N+3" as applying to all three of the cases that were presented. In this way, the N stands not necessarily for one particular instance as an unknown, but as a variable in a description of the relationship between the pairs of numbers.

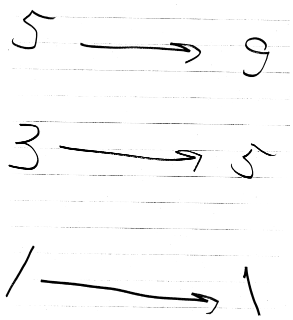
The transition from understanding letters as unknowns, to understanding them as variables is notoriously difficult, even for adolescents. However, in the present activity, with a simple additive function, they follow the idea with little trouble, as Melissa shows:

> Melissa: Yeah, because you have eleven. Well. Lets just say you have ten, then you add three more and which…I mean, you have eleven, then you have three more, equal fourteen. And say if you did it with twelve, that equals fifteen.
> David: Hold on. Twelve becomes fifteen. Yes. That's correct.
> Melissa: And, like the higher we go, the higher the numbers get.
> David: That's right. So, could you do a hundred?
> Melissa: Yeah.
> Students: A hundred and three.
> David: That's great.
> Melissa: And if you do a thousand, it's a thousand and three.

Melissa first offers the cases of 11, then 12, and then attempts to generalize: "the higher we go, the higher the numbers get". This suggests that she is referring to two *sets of numbers*, the numbers chosen ("the higher we go") as well as the numbers that emerge from applying the rule ("the higher the numbers get"). The numbers are connected one to one as ordered pairs; for each number-input there is a respective output number. But she can also mentally scan the diverse cases in an ordered fashion and think about how variations in input are related to variations in output. The numbers co-vary according to a remarkably simple pattern: as input values increase, the output values increase, with the constraint that the latter are in every case precisely three units more than the former.

In response to the children's demand to give them "something harder", David wrote the following number pairs (see Figure 3), one by one, and asked the children to guess the rule he was using.

Figure 3: Input and outputs for n→2n-1



The class discusses the possible rule that generates the output numbers from the input numbers:

David: [As he writes 3 and 5 in the second row] OK. If I give you a three, you've got to get a five out. You think you still know? You think you know, Michael?

Michael: Yeah.

David: If I gave you an n [writes "n→" above the number pairs] then what, OK…

Michael: For the first one…

David: For the first one, how do you get from five to nine?

Michael: Add four.

David: You add four. And if I add four to three?

Students: No.

David: You could've been right. Cause that's one way to get from five to nine (adding 4). However, this rule, it can't be that rule cause it didn't work for the second one (from 3 to 5). Because if I added four, this would become seven, and it became five. Let me give you another example. If I give you a one, do you know what you're gonna get from this?

James: Oh, I know!

David: James. Let's see if he's got it.

James: You have to add two?

David: You add two? So if you add two to five you get how much?

Student: Nine.

David: No, you don't get nine.

Student: Seven.

David: Actually, it's not as hard. If I give you a one, you have to get out a one.

Student: Oh, one times one equals one.

David: One times one would be one. But five times five isn't nine.

Jessica: Sara knows!

David: Sara, give us a, clarify for us.

Sara: Two times that number minus one.

David: Wow! Wow! Sara, come here, write this up here. Write it up here, if you can generalize it.

Sara: [Writes "x 2 – 1" above the second column of numbers, following "n→"].
David: Write the n in front so we remember, n times 2 minus one.
Sara: [Completes the notation "n→n x 2 – 1"].
David: Have you guys got this figured out? Did you see what she did? So you have to use which times table, Sara? This is really something!
Students: Harder, harder.
David: Pardon me?
Students: Harder, harder.

Children who proposed additive rules such as “n→ n+2” may have been thinking only about particular cases. However Sara’s rule, “n→n x 2 –1”, does not merely describe the relationship between two known values but encompasses each of the cases listed. It is remarkable that she does this in the very lesson in which mapping notation is introduced.

Subsequent discussion in the same lesson showed that only Sara and a small number of her peers were able to generate such “linear function” rules from multiple instances. However, students in the class understood how the rule could account for each of the individual instances. In fact, once a student would propose a rule, other students, including those who did not themselves generate such rules, eagerly volunteered to argue whether the rule worked for the set of instances or just for isolated cases. In this sense they were able to begin to think functionally and to make use of functional notation. Furthermore, when dealing with simpler, additive functions, such as n→ n+3, most students were able to meaningfully generate and use algebraic notation for functions.

Students may not quickly learn to identify linear functions underlying data. Despite this, and perhaps *because* of this, linear functions can begin to be explored as extensions to students’ work with multiplication tables. Further, even though not all third grade students will initially identify and represent the functional relationships underlying data tables, they can learn significant things in the resulting discussions and slowly work functional notation into their arsenal of representational tools.

Discussion

In introducing a data table with number of items and prices, we found that the students could correctly fill in the tables, but they did so with a minimal of thought about the invariant relationship between the values in the first and second columns. Several changes were made in the structure of the table and the purposes of the activities to discourage students from working on each column as if it were unrelated to the column next to it.

A guess-my-rule game helped students break away from the isolated column strategies they had been using. One important feature of the game was that there was no discernible downward progression from row to row. This seemed to deter students from viewing the data from a within-measure perspective.

It surprised us that the 9-year-old children were content to look for patterns and functional relations among pure numbers devoid of quantitative reference. They did not need concrete materials to support their reasoning about numerical relations and could even deal with notations of an algebraic nature. In fact, algebraic notation seemed to help them move from computational aspects to generalizations about how two sets of values are interrelated.

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